

UNIVERSAL DEFORMATION RINGS OF STRINGS MODULES OVER A CERTAIN SYMMETRIC TAME ALGEBRA

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ABSTRACT. Let \mathbb{k} be an algebraically closed field, let Λ be a finite dimensional \mathbb{k} -algebra and let V be a Λ -module with stable endomorphism ring isomorphic to \mathbb{k} . If Λ is self-injective then V has a universal deformation ring $R(\Lambda, V)$, which is a complete local commutative Noetherian \mathbb{k} -algebra with residue field \mathbb{k} . Moreover, if Λ is also a Frobenius \mathbb{k} -algebra then $R(\Lambda, V)$ is stable under syzygies. We use these facts to determine the universal deformation rings of string $\Lambda_{\bar{r}}$ -modules whose stable endomorphism ring is isomorphic to \mathbb{k} , where $\Lambda_{\bar{r}}$ is a symmetric special biserial \mathbb{k} -algebra that has quiver with relations depending on the four parameters $\bar{r} = (r_0, r_1, r_2, k)$ with $r_0, r_1, r_2 \geq 2$ and $k \geq 1$. Universal deformation rings and Frobenius algebras and Stable endomorphism rings and Special biserial algebras [2000]16G10 and 16G20 and 20C20

1. INTRODUCTION

Let \mathbb{k} be a field of arbitrary characteristic, and let denote by $\hat{\mathcal{C}}$ the category of all complete local commutative Noetherian \mathbb{k} -algebras with residue field \mathbb{k} . Suppose that Λ is a fixed finite dimensional \mathbb{k} -algebra and let V be a finitely generated Λ -module. Let R be an arbitrary object in $\hat{\mathcal{C}}$. A *lift* (M, ϕ) of V over R is a finitely generated $R \otimes_{\mathbb{k}} \Lambda$ -module M that is free over R together with an isomorphism of Λ -modules $\phi : \mathbb{k} \otimes_R M \rightarrow V$. If Λ is self-injective and the stable endomorphism ring of V is isomorphic to \mathbb{k} , then there exists a particular object $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ and a lift $(U(\Lambda, V), \phi_{U(\Lambda, V)})$ of V over $R(\Lambda, V)$, which is universal with respect to all isomorphism classes of lifts of V over such \mathbb{k} -algebras R (see [10] and §2). The ring $R(\Lambda, V)$ and the isomorphism class of the lift $(U(\Lambda, V), \phi_{U(\Lambda, V)})$ are respectively called the *universal deformation ring* and the *universal deformation* of V . Traditionally, universal deformations rings are studied when Λ is equal to a group algebra $\mathbb{k}G$, where G is a finite group and \mathbb{k} has positive characteristic p (see e.g., [3, 4, 5, 6, 7, 8, 9]). In particular, it was proved by F. M. BLEHER and T. CHINBURG in [4] that if V is a finitely generated $\mathbb{k}G$ -module whose stable endomorphism ring is isomorphic to \mathbb{k} , then V has a universal deformation ring $R(G, V)$. Observe that $\mathbb{k}G$ is an example of a self-injective \mathbb{k} -algebra (see e.g., [2, Prop. 3.1.2]). This approach has recently led to the solution of various open problems, e.g., the construction of representations whose universal deformation rings are not local complete intersections (see [3, 5, 6]). Universal deformation rings of modules over more general finite dimensional algebras have been studied by many authors in different contexts (see e.g., [14, 18] and their references). The main motivation of this article is that sophisticated results from representation theory of finite dimensional algebras, such as Auslander-Reiten quivers, stable equivalences, and combinatorial description of modules can be used to arrive at a deeper understanding of universal deformation rings.

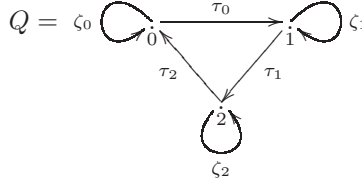
In this article, we assume that \mathbb{k} is algebraically closed and consider the basic \mathbb{k} -algebra

$$(1) \quad \Lambda_{\bar{r}} = \mathbb{k}Q/I_{\bar{r}}$$

Key words and phrases. Universal deformation rings and Frobenius algebras and Stable endomorphism rings.

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where $\bar{r} = (r_0, r_1, r_2, k)$ with $r_0, r_1, r_2 \geq 2, k \geq 1$, Q is the quiver



and $I_{\bar{r}}$ is the ideal of the path algebra $\mathbb{k}Q$ generated by the relations

$$\{\tau_0\zeta_0, \zeta_1\tau_0, \tau_1\zeta_1, \zeta_2\tau_1, \tau_2\zeta_2, \zeta_0\tau_2, \zeta_0^{r_0} - (\tau_2\tau_1\tau_0)^k, \zeta_1^{r_1} - (\tau_0\tau_2\tau_1)^k, \zeta_2^{r_2} - (\tau_1\tau_0\tau_2)^k\}.$$

The algebra $\Lambda_{\bar{r}}$ is among the class of algebras of dihedral type, which were introduced by K. ERDMANN in [13] to classify all tame blocks of group algebras of finite groups with dihedral defect groups up to Morita equivalence. However, $\Lambda_{\bar{r}}$ is not Morita equivalent to a block of a group algebra (see [13, Lemma IX.5.4]). Since $\Lambda_{\bar{r}}$ is a special biserial algebra, all the non-projective indecomposable $\Lambda_{\bar{r}}$ -modules can be described combinatorially as so-called strings and bands modules as introduced in [11] (see also §3.1). We denote by $\Gamma_s(\Lambda_{\bar{r}})$ the stable Auslander-Reiten quiver of $\Lambda_{\bar{r}}$. The components of $\Gamma_s(\Lambda_{\bar{r}})$ consisting in string modules are two 3-tubes and infinitely many components of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$. The components consisting of band modules are infinitely many 1-tubes.

In [10], the particular case $\bar{r} = (2, 2, 2, 1)$ has been considered. In particular, there are exactly three components \mathfrak{C} of $\Gamma_s(\Lambda_{\bar{r}})$ of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$, which each contain a simple $\Lambda_{\bar{r}}$ -module. If \mathfrak{C} is such a component then $\Omega(\mathfrak{C}) = \mathfrak{C}$ and there are exactly three Ω -orbits of $\Lambda_{(2,2,2,1)}$ -modules in \mathfrak{C} whose stable endomorphism ring is isomorphic to \mathbb{k} ; the universal deformation rings are either isomorphic to \mathbb{k} , or to $\mathbb{k}[[t]]/(t^2)$, or to $\mathbb{k}[[t]]$ (see [10, Prop. 3.9]). Moreover, if \mathfrak{T} is one 3-tube of $\Gamma_s(\Lambda_{(2,2,2,1)})$ then $\Omega(\mathfrak{T})$ is the other 3-tube and there are exactly three Ω -orbits of $\Lambda_{(2,2,2,1)}$ -modules in \mathfrak{T} whose stable endomorphism ring is isomorphic to \mathbb{k} ; the universal deformation rings are either isomorphic to \mathbb{k} or to $\mathbb{k}[[t]]$ (see [10, Prop. 3.11]).

In this article, we let $\bar{r} = (r_0, r_1, r_2, k)$ with $r_0, r_1, r_2 \geq 2$ and $k \geq 1$ be arbitrary. We study the two 3-tubes and the components \mathfrak{C} of $\Gamma_s(\Lambda_{\bar{r}})$ of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$ containing a module whose endomorphism ring is isomorphic to \mathbb{k} . Our goal is to investigate how universal deformation rings change when inflating modules from $\Lambda_{(r_0, r_1, r_2, k)}$ to $\Lambda_{(r'_0, r'_1, r'_2, k')}$, where $\Lambda_{(r'_0, r'_1, r'_2, k')}$ surjects onto $\Lambda_{(r_0, r_1, r_2, k)}$ when $r'_0 \geq r_0, r'_1 \geq r_1, r'_2 \geq r_2, k' \geq k$.

If M and N are two indecomposable $\Lambda_{\bar{r}}$ -modules belonging to the same component of $\Gamma_s(\Lambda_{\bar{r}})$, we say that N is a *successor* of M provided that there exists an irreducible homomorphism $M \rightarrow N$. Throughout this article, we identify the vertices of the quiver Q with elements of the cyclic group with three elements $\mathbb{Z}/3$.

A summary of the main results concerning $\Lambda_{\bar{r}} = \mathbb{k}Q/I_{\bar{r}}$ is as follows (cf. [10, Prop. 3.9, Prop. 3.11]); for more precise statements, see Propositions 4.1, 4.4, 4.5 and 4.6.

Theorem 1.1. *Let $\Lambda_{\bar{r}}$ be as in (1) where $\bar{r} = (r_0, r_1, r_2, k)$ with $r_0, r_1, r_2 \geq 2$ and $k \geq 1$, and let $\Gamma_s(\Lambda_{\bar{r}})$ denote the stable Auslander-Reiten quiver of $\Lambda_{\bar{r}}$.*

- (i) *If \mathfrak{T} is one of two the 3-tubes then $\Omega(\mathfrak{T})$ is the other 3-tube. There are exactly three Ω -orbits of modules in $\mathfrak{T} \cup \Omega(\mathfrak{T})$ whose stable endomorphism ring is isomorphic to \mathbb{k} . If X_0 is a module that belongs to the boundary of \mathfrak{T} , then these three Ω -orbits are represented by X_0 , by a successor X_1 of X_0 , and by a successor X_2 of X_1 that does not lie in the Ω -orbit of X_0 . The universal deformation rings are*

$$R(\Lambda_{\bar{r}}, X_0) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, X_1) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, X_2) \cong \mathbb{k}[[t]].$$

- (ii) *There are exactly three distinct components $\mathfrak{A}_0, \mathfrak{A}_1$ and \mathfrak{A}_2 of $\Gamma_s(\Lambda_{\bar{r}})$ of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$, which each contain exactly one simple $\Lambda_{\bar{r}}$ -module. For all $i \in \{0, 1, 2\} \bmod 3$, the component \mathfrak{A}_i is Ω -stable if and only if $r_i = 2$ and there are exactly three Ω -orbits of modules in $\mathfrak{A}_i \cup \Omega(\mathfrak{A}_i)$ whose stable endomorphism ring is isomorphic to \mathbb{k} . If for all $i \in \{0, 1, 2\} \bmod 3$, $U_{i,0}$ denotes the unique simple module lying in \mathfrak{A}_i , then these three Ω -orbits are represented by $U_{i,0}$, by a successor $U_{i,1}$*

of $U_{i,0}$, and by a successor $U_{i,2}$ of $U_{i,1}$ that does not lie in the Ω -orbit of $U_{i,0}$. The universal deformation rings are

$$R(\Lambda_{\bar{r}}, U_{i,0}) \cong \mathbb{k}[[t]]/(t^{r_i}), \quad R(\Lambda_{\bar{r}}, U_{i,1}) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, U_{i,2}) \cong \mathbb{k}[[t]].$$

- (iii) There are three distinct components \mathfrak{B}_0 , \mathfrak{B}_1 and \mathfrak{B}_2 of $\Gamma_s(\Lambda_{\bar{r}})$ of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$ that contain exactly a module of length 1 whose endomorphism ring is isomorphic to \mathbb{k} . Let $i \in \{0, 1, 2\} \bmod 3$ and let $V_{i,0}$ be a module of minimal length in \mathfrak{B}_i . If $k = 1$ then $\mathfrak{B}_i = \Omega(\mathfrak{A}_{i+2})$, where \mathfrak{A}_{i+2} is as in (ii). In particular, \mathfrak{B}_i is Ω -stable if and only if $k = 1$ and $r_{i+2} = 2$. There are exactly three Ω -orbits of modules in $\mathfrak{B}_i \cup \Omega(\mathfrak{B}_i)$ whose stable endomorphism ring is isomorphic to \mathbb{k} . These three Ω -orbits are represented by $V_{i,0}$, by a successor $V_{i,1}$ of $V_{i,0}$, and by a successor $V_{i,-1}$ of $V_{i,0}$ that does not lie in the Ω -orbit of $V_{i,1}$. If $k = 1$ then the universal deformation rings are

$$R(\Lambda_{\bar{r}}, V_{i,0}) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, V_{i,1}) \cong \mathbb{k}[[t]]/(t^{r_{i+2}}), \quad R(\Lambda_{\bar{r}}, V_{i,-1}) \cong \mathbb{k}[[t]].$$

If $k \geq 2$ then the universal deformation rings are

$$R(\Lambda_{\bar{r}}, V_{i,0}) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, V_{i,1}) \cong \mathbb{k}[[t]], \quad R(\Lambda_{\bar{r}}, V_{i,-1}) \cong \mathbb{k}[[t]].$$

- (iv) If $k \geq 2$ then there are three distinct components \mathfrak{C}_0 , \mathfrak{C}_1 and \mathfrak{C}_2 of $\Gamma_s(\Lambda_{\bar{r}})$ of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$, which each contain a module of length 2 whose endomorphism ring is isomorphic to \mathbb{k} . Let $i \in \{0, 1, 2\} \bmod 3$ and let $W_{i,0}$ be a module of minimal length in \mathfrak{B}_i . For all $i \in \{0, 1, 2\} \bmod 3$, the component \mathfrak{C}_i is Ω -stable if and only if $k = 2$, and there are exactly three Ω -orbits of modules in \mathfrak{C}_i whose stable endomorphism ring is isomorphic to \mathbb{k} . These three Ω -orbits are represented by $W_{i,0}$, by a successor $W_{i,-1}$ of $W_{i,0}$, and by a successor $W_{i,-2}$ of $W_{i,-1}$ that does not lie in the Ω -orbit of $W_{i,0}$. The universal deformation rings are

$$R(\Lambda_{\bar{r}}, W_{i,0}) \cong \mathbb{k}[[t]]/(t^k), \quad R(\Lambda_{\bar{r}}, W_{i,-1}) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, W_{i,-2}) \cong \mathbb{k}[[t]].$$

This article is organized as follows. In §2, we recall the definitions of deformations and universal deformation rings and summarize some of their properties. In §3, we give a precise description of string modules for $\Lambda_{\bar{r}}$, describe the components of $\Gamma_s(\Lambda_{\bar{r}})$ that contain string modules using hooks and co-hooks (see [11]), and give a description of the homomorphisms between strings modules as determined in [15]. Moreover, we describe the indecomposable projective modules for $\Lambda_{\bar{r}}$ and classify all $\Lambda_{\bar{r}}$ -modules with endomorphism ring isomorphic to \mathbb{k} (see Proposition 3.1). In §4, we prove Theorem 1.1.

See e.g., [1, 2, 13] for further information about basic concepts from representation theory of finite dimensional algebras, such as the definition and properties of the syzygy functor Ω and the definition of the Auslander-Reiten quiver of an arbitrary Artinian algebra Λ .

2. UNIVERSAL DEFORMATION RINGS

Let \mathbb{k} be a field of arbitrary characteristic and denote by $\hat{\mathcal{C}}$ the category of all complete local commutative Noetherian \mathbb{k} -algebras with residue field \mathbb{k} . Note that the morphisms in $\hat{\mathcal{C}}$ are continuous \mathbb{k} -algebra homomorphisms that induce the identity map on \mathbb{k} . Suppose that Λ is a finite dimensional \mathbb{k} -algebra and V is a fixed finitely generated Λ -module. We denote by $\text{End}_{\Lambda}(V)$ (respectively, by $\underline{\text{End}}_{\Lambda}(V)$) the endomorphism ring (respectively, the stable endomorphism ring) of V . Let R be an arbitrary object in $\hat{\mathcal{C}}$. A *lift* (M, ϕ) of V over R is a finitely generated $R \otimes_{\mathbb{k}} \Lambda$ -module M that is free over R together with an isomorphism of Λ -modules $\phi : \mathbb{k} \otimes_R M \rightarrow V$. Two lifts (M, ϕ) and (M', ϕ') over R are *isomorphic* if there exists an $R \otimes_{\mathbb{k}} \Lambda$ -module isomorphism $f : M \rightarrow M'$ such that $\phi' \circ (\text{id}_{\mathbb{k}} \otimes f) = \phi$, where $\text{id}_{\mathbb{k}}$ denotes the identity map on \mathbb{k} . If (M, ϕ) is a lift of V over R we denote by $[M, \phi]$ its isomorphism class and say that $[M, \phi]$ is a *deformation* of V over R . We denote by $\text{Def}_{\Lambda}(V, R)$ the set of all deformations of V over R . The *deformation functor* over V is the covariant functor $\hat{F}_V : \hat{\mathcal{C}} \rightarrow \text{Sets}$ defined as follows: for all objects R in $\hat{\mathcal{C}}$ define $\hat{F}_V(R) = \text{Def}_{\Lambda}(V, R)$ and for all morphisms $\alpha : R \rightarrow R'$ in $\hat{\mathcal{C}}$ let $\hat{F}_V(\alpha) : \text{Def}_{\Lambda}(V, R) \rightarrow \text{Def}_{\Lambda}(V, R')$ be defined as $\hat{F}_V(\alpha)([M, \phi]) = [R' \otimes_{R, \alpha} M, \phi_{\alpha}]$, where $\phi_{\alpha} : \mathbb{k} \otimes_{R'} (R' \otimes_{R, \alpha} M) \rightarrow V$ is the composition of Λ -module isomorphisms

$$\mathbb{k} \otimes_{R'} (R' \otimes_{R, \alpha} M) \cong \mathbb{k} \otimes_R M \xrightarrow{\phi} V.$$

Suppose there exists an object $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ of V over $R(\Lambda, V)$ with the following property. For each R in $\hat{\mathcal{C}}$ and for all lifts M of V over R there exists a morphism $v : R(\Lambda, V) \rightarrow R$ in $\hat{\mathcal{C}}$ such that

$$\hat{F}_V(v)[U(\Lambda, V), \phi_{U(\Lambda, V)}] = [M, \phi],$$

and moreover v is unique if R is the ring of dual numbers $\mathbb{k}[[t]]/(t^2)$. Then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ are respectively called the *versal deformation ring* and *versal deformation* of V . If the morphism v is unique for all R in $\hat{\mathcal{C}}$ and lifts (M, ϕ) of V over R , then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ are respectively called the *universal deformation ring* and the *universal deformation* of V . In other words, the universal deformation ring $R(\Lambda, V)$ represents the deformation functor \hat{F}_V in the sense that \hat{F}_V is naturally isomorphic to the Hom functor $\text{Hom}_{\hat{\mathcal{C}}}(R(\Lambda, V), -)$. Using Schlessinger's criteria [17, Thm. 2.11] and using methods similar to those in [16], it is straightforward to prove that the deformation functor \hat{F}_V is continuous, that every finitely generated Λ -module V has a versal deformation ring and that this versal deformation is universal provided that the endomorphism ring $\text{End}_{\Lambda}(V)$ is isomorphic to \mathbb{k} (see [10, Prop. 2.1]).

Recall that the \mathbb{k} -algebra Λ is said to be self-injective if the regular left Λ -module ${}_{\Lambda}\Lambda$ is injective and that Λ is called a Frobenius algebra provided that the right Λ -modules Λ_{Λ} and $({}_{\Lambda}\Lambda)^* = \text{Hom}_{\mathbb{k}}({}_{\Lambda}\Lambda, \mathbb{k})$ are isomorphic. Recall also that Λ is said to be a symmetric algebra provided that Λ is a Frobenius algebra and there exists a non-degenerate associative bilinear form $\theta : \Lambda \times \Lambda \rightarrow \mathbb{k}$ with $\theta(a, b) = \theta(b, a)$ for all $a, b \in \Lambda$. By [12, Prop. 9.9], every Frobenius algebra is self-injective.

Remark 2.1. If Λ is self-injective and (M, ϕ) is a lift of V over an object R in $\hat{\mathcal{C}}$ with $\text{End}_{\Lambda}(V) \cong \mathbb{k}$, then the deformation $[M, \phi]$ does not depend on the particular choice of the Λ -module isomorphism. More precisely, if $f : M \rightarrow M'$ is an $R \otimes_{\mathbb{k}} \Lambda$ -module isomorphism with (M', ϕ') a lift of V over R , then there exists an $R \otimes_{\mathbb{k}} \Lambda$ -module isomorphism $\bar{f} : M \rightarrow M'$ such that $\phi' \circ (\text{id}_{\mathbb{k}} \otimes_R f) = \phi$. In other words, $[M, \phi] = [M', \phi']$ in $\hat{F}_V(R) = \text{Def}_{\Lambda}(V, R)$ (see [10, Thm. 2.6]).

We denote the first syzygy of V by ΩV , i.e., ΩV is the kernel of a projective cover $P_V \rightarrow V$, (see e.g., [1, pp. 124-126]).

Example 2.2. Let G be a finite group and consider the group algebra $\mathbb{k}G$, which is a self-injective \mathbb{k} -algebra (see e.g., [2, Prop. 3.1.2] and [12, Prop. 9.6]). It was proved in [4] that if V is a finitely generated $\mathbb{k}G$ -module whose stable endomorphism ring is isomorphic to \mathbb{k} then V has a universal deformation ring $R(\mathbb{k}G, V)$. Moreover, the stable endomorphism ring of ΩV is also isomorphic to \mathbb{k} and the universal deformation rings $R(\mathbb{k}G, V)$ and $R(\mathbb{k}G, \Omega V)$ of V and ΩV , respectively, are isomorphic.

The following result generalizes the properties of universal deformation rings mentioned in Example 2.2 to arbitrary Frobenius \mathbb{k} -algebras (see [10, Thm. 2.6]).

Theorem 2.3. *Let Λ be a finite dimensional self-injective \mathbb{k} -algebra, and suppose that V is a finitely generated Λ -module whose stable endomorphism ring $\text{End}_{\Lambda}(V)$ is isomorphic to \mathbb{k} .*

- (i) *The module V has a universal deformation ring $R(\Lambda, V)$.*
- (ii) *If P is a finitely generated projective Λ -module, then $\text{End}_{\Lambda}(V \oplus P) \cong \mathbb{k}$ and $R(\Lambda, V) \cong R(\Lambda, V \oplus P)$.*
- (iii) *If Λ is also a Frobenius algebra, then $\text{End}_{\Lambda}(\Omega V) \cong \mathbb{k}$ and $R(\Lambda, V) \cong R(\Lambda, \Omega V)$.*

3. SOME REMARKS ABOUT THE REPRESENTATION THEORY OF $\Lambda_{\bar{r}}$ AND CLASSIFICATION OF $\Lambda_{\bar{r}}$ -MODULES WHOSE ENDOMORPHISM RING IS ISOMORPHIC TO \mathbb{k}

For the remainder of this article, let \mathbb{k} be an algebraically closed field of arbitrary characteristic and let $\Lambda_{\bar{r}} = \mathbb{k}Q/I_{\bar{r}}$ as in (1). We identify the vertices of Q with elements of $\mathbb{Z}/3$ (the cyclic group of three elements).

The algebra $\Lambda_{\bar{r}}$ is one of the algebras of dihedral type studied by K. ERDMANN in [13]. In particular, $\Lambda_{\bar{r}}$ is a symmetric \mathbb{k} -algebra. However, by [13, Lemma IX.5.4], $\Lambda_{\bar{r}}$ is not Morita equivalent to a block of a group algebra. Since $\Lambda_{\bar{r}}$ is a special biserial algebra, all the non-projective indecomposable $\Lambda_{\bar{r}}$ -modules can be described combinatorially as so-called strings and bands modules (see [11]). In this article, we are only concerned about these string modules, which are described as follows.

3.1. String modules for $\Lambda_{\bar{r}}$. Given each arrow $\zeta_0, \tau_0, \zeta_1, \tau_1, \zeta_2, \tau_2$ of Q , we define a formal inverse by $\zeta_0^{-1}, \tau_0^{-1}, \zeta_1^{-1}, \tau_1^{-1}, \zeta_2^{-1}, \tau_2^{-1}$, respectively. Let $\mathbf{s}(\zeta_0) = 0 = \mathbf{s}(\zeta_0^{-1})$, $\mathbf{s}(\tau_0) = 0 = \mathbf{s}(\tau_0^{-1})$, $\mathbf{s}(\zeta_1) = 1 = \mathbf{s}(\zeta_1^{-1})$, $\mathbf{s}(\tau_1) = 1 = \mathbf{s}(\tau_1^{-1})$, $\mathbf{s}(\zeta_2) = 2 = \mathbf{s}(\zeta_2^{-1})$ and $\mathbf{s}(\tau_2) = 2 = \mathbf{s}(\tau_2^{-1})$. Let $\mathbf{e}(\zeta_0) = 0 = \mathbf{e}(\zeta_0^{-1})$, $\mathbf{e}(\tau_0) = 1 = \mathbf{e}(\tau_0^{-1})$, $\mathbf{e}(\zeta_1) = 1 = \mathbf{e}(\zeta_1^{-1})$, $\mathbf{e}(\tau_1) = 2 = \mathbf{s}(\tau_2^{-1})$, $\mathbf{e}(\zeta_2) = 2 = \mathbf{e}(\zeta_2^{-1})$ and $\mathbf{e}(\tau_2) = 0 = \mathbf{e}(\tau_0^{-1})$. By a *word* of length $n \geq 1$ we mean a sequence $w_n \cdots w_1$, where the w_j is either an arrow or a formal inverse of an arrow and where $\mathbf{s}(w_{j+1}) = \mathbf{e}(w_j)$ for $1 \leq j \leq n-1$. We define $(w_n \cdots w_1)^{-1} = w_1^{-1} \cdots w_n^{-1}$, $\mathbf{s}(w_n \cdots w_1) = \mathbf{s}(w_1)$ and $\mathbf{e}(w_n \cdots w_1) = \mathbf{e}(w_n)$. If $i \in \{0, 1, 2\} \bmod 3$ is a vertex of Q , we define an empty word $\mathbb{1}_i$ of length zero with $\mathbf{e}(\mathbb{1}_i) = i = \mathbf{s}(\mathbb{1}_i)$ and $(\mathbb{1}_i)^{-1} = \mathbb{1}_i$. For all $i \in \{0, 1, 2\} \bmod 3$ there exists a path in Q of length 3 starting and ending at i , namely

$$(2) \quad \underline{\zeta}_i = \tau_{i+2}\tau_{i+1}\tau_i.$$

Denote by \mathcal{W} the set of all words and let

$$J = \{\zeta_0^{r_0}, \zeta_1^{r_2}, \zeta_2^{r_3}, \tau_0\zeta_0, \zeta_1\tau_0, \tau_1\zeta_1, \zeta_2\tau_1, \tau_2\zeta_2, \zeta_0\tau_2, \underline{\zeta}_0^k, \underline{\zeta}_1^k, \underline{\zeta}_2^k\}.$$

Let \sim be the equivalence relation on \mathcal{W} defined by $w \sim w'$ if and only if $w = w'$ or $w^{-1} = w'$. A *string* is a representative C of an equivalence class under the relation \sim where either $C = \mathbb{1}_i$ for some vertex i of Q , or $C = w_n \cdots w_1$ with $n \geq 1$ and $w_j \neq w_{j+1}^{-1}$ for $1 \leq j \leq n-1$ and no sub-word of C or its formal inverse belong to J . If C is a string such that $\mathbf{s}(C) = \mathbf{e}(C)$, then we let $C^0 = \mathbb{1}_{\mathbf{e}(C)}$. If $C = w_n \cdots w_1$ and $D = v_m \cdots v_1$ are strings of length $n, m \geq 1$, respectively, we say that the composition of C and D is defined provided that $w_n \cdots w_1 v_m \cdots v_1$ is a string, and write $CD = w_n \cdots w_1 v_m \cdots v_1$; we say that the composition of C with $\mathbb{1}_i$ is defined provided $\mathbf{s}(C) = i$ (respectively, $\mathbf{e}(C) = i$), and in this case we have $C\mathbb{1}_i \sim C$ (respectively, $\mathbb{1}_i C \sim C$). In particular, if $C = w_n \cdots w_1$ is a string of length $n \geq 1$ then $C \sim w_n \cdots w_{j+1} \mathbb{1}_{\mathbf{e}(w_j)} w_j \cdots w_1$ for all $1 \leq j \leq n-1$. If $C = w_n \cdots w_1$ is a string of length $n \geq 1$ then there exists an indecomposable $\Lambda_{\bar{r}}$ -module $M[C]$, called the *string module* corresponding to the string representative C , which can be described as follows. There is an ordered \mathbb{k} -basis $\{z_0, z_1, \dots, z_n\}$ of $M[C]$ such that the action of $\Lambda_{\bar{r}}$ on $M[C]$ is given by the following representation $\varphi_C : \Lambda_{\bar{r}} \rightarrow \text{Mat}(n+1, \mathbb{k})$. Let $\mathbf{v}(j) = \mathbf{e}(w_j)$ for $0 \leq j \leq n-1$ and $\mathbf{v}(n) = \mathbf{s}(w_n)$. Then for each vertex $i \in \{0, 1, 2\} \bmod 3$ and for each arrow $\zeta \in \{\zeta_0, \tau_0, \zeta_1, \tau_1, \zeta_2, \tau_2\}$ in Q and for all $0 \leq j \leq n$ define

$$(3) \quad \varphi_C(i)(z_j) = \begin{cases} z_j, & \text{if } \mathbf{v}(j) = i \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \varphi_C(\zeta)(z_j) = \begin{cases} z_{j-1}, & \text{if } w_j = \zeta \\ z_{j+1}, & \text{if } w_{j+1} = \zeta^{-1} \\ 0, & \text{otherwise} \end{cases}$$

We call φ_C the *canonical representation* and $\{z_0, z_1, \dots, z_n\}$ a *canonical \mathbb{k} -basis* for $M[C]$ relative to the string representative C . Note that $M[C] \cong M[C^{-1}]$. If $C = \mathbb{1}_i$ with $i \in \{0, 1, 2\} \bmod 3$ then $M[C]$ is the simple $\Lambda_{\bar{r}}$ -module corresponding to vertex i . We denote the simple $\Lambda_{\bar{r}}$ -modules corresponding to the vertices 0, 1 and 2 of Q by $M[\mathbb{1}_0]$, $M[\mathbb{1}_1]$ and $M[\mathbb{1}_2]$, respectively.

3.2. The stable Auslander-Reiten quiver of $\Lambda_{\bar{r}}$. We denote by $\Gamma_s(\Lambda_{\bar{r}})$ the stable Auslander-Reiten quiver of $\Lambda_{\bar{r}}$ (see [1, VII]). For all $i \in \{0, 1, 2\} \bmod 3$ there exists a path in Q of length 3 starting and ending at i , namely

$$(4) \quad \underline{\zeta}_i = \tau_{i+2}\tau_{i+1}\tau_i.$$

The components \mathfrak{C} of $\Gamma_s(\Lambda_{\bar{r}})$ consisting of string $\Lambda_{\bar{r}}$ -modules are two 3-tubes and infinitely many non-periodic components of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$. In the following, we describe the irreducible morphism between string $\Lambda_{\bar{r}}$ -modules.

Assume $C = w_n w_{n-1} \cdots w_1$ with $n \geq 1$ is a string. We say that C is *directed* if all w_j are arrows and we say that C is a *maximal directed string* if C is directed and if for any arrow ζ in Q , $\zeta C \in J$. Let \mathcal{M} be the set of all maximal directed strings, i.e.,

$$\mathcal{M} = \{\zeta_0^{r_0-1}, \zeta_1^{r_1-1}, \zeta_2^{r_2-1}, \underline{\zeta}_0^{k-1}\tau_2\tau_1, \underline{\zeta}_1^{k-1}\tau_0\tau_2, \underline{\zeta}_2^{k-1}\tau_1\tau_0\}.$$

Let C be a string. We say that C *starts on a peak* (respectively, *starts in a deep*) provided that there is no arrow ζ in Q such that $C\zeta$ (respectively, $C\zeta^{-1}$) is a string; we also say that C *ends in a peak* (respectively,

ends in a deep) provided that there is no arrow γ in Q such that $\gamma^{-1}C$ (respectively, γC) is a string. If C is a string not starting on a peak (respectively, not starting in a deep), say $C\zeta$ (respectively, $C\zeta^{-1}$) is a string for some arrow ζ then there is a unique directed string $D \in \mathcal{M}$ such that $C_h = C\zeta D^{-1}$ (respectively, $C_c = C\zeta^{-1}D$) is a string. We say C_h (respectively, C_c) is obtained from C by adding a *hook* (respectively, a *co-hook*) on the right side. Dually, if C is a string not ending on a peak (respectively, not ending in a deep), say $\gamma^{-1}C$ (respectively, γC) is a string for some arrow γ in Q then there is a unique directed string $E \in \mathcal{M}$ such that ${}_h C = E\gamma^{-1}C$ (respectively, ${}_c C = E^{-1}\gamma C$) is a string. We say ${}_h C$ (respectively, ${}_c C$) is obtained from C by adding a *hook* (respectively, a *co-hook*) on the left side. By [11], all irreducible morphisms between string modules are either canonical injections $M[C] \rightarrow M[C_h]$, $M[C] \rightarrow M[{}_h C]$, or canonical projections $M[C_c] \rightarrow M[C]$, $M[{}_c C] \rightarrow M[C]$. Suppose $M[C]$ is a string module of minimal length such that $M[C]$ belongs to a component \mathfrak{C} of $\Gamma_s(\Lambda_{\bar{r}})$ of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$. Since none of the projective $\Lambda_{\bar{r}}$ -modules is uniserial then

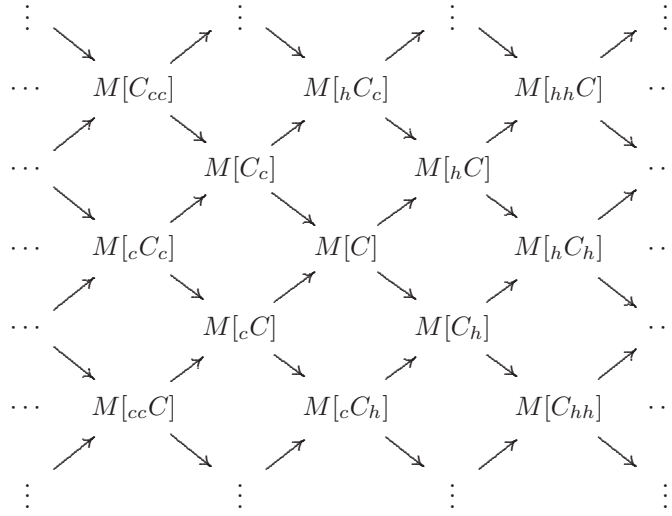


FIGURE 1. The stable Auslander-Reiten component near $M[C]$.

near $M[C]$ the component \mathfrak{C} looks as in Figure 1.

3.3. Homomorphisms between string modules for $\Lambda_{\bar{r}}$. Let S and T be strings for $\Lambda_{\bar{r}}$. Suppose C is a substring of both S and T such that the following conditions (i) and (ii) are satisfied.

- (i) $S \sim BCD$, where B is a substring which is either of length zero or $B = B'\zeta$ for an arrow ζ , and D is a substring which is either of length zero or $D = \gamma^{-1}D'$ for an arrow γ , i.e., $S \sim B' \xleftarrow{\tau} C \xrightarrow{\varphi} D'$.
- (ii) $T \sim ECF$, where E is a substring which is either of length zero or $E = E'\epsilon^{-1}$ for an arrow ϵ , and F is a substring which is either of length zero or $F = \mu F'$ for an arrow μ , i.e., $T \sim E' \xrightarrow{\epsilon} C \xleftarrow{\mu} F'$.

Then by [15] there exists a composition of $\Lambda_{\bar{r}}$ -module homomorphisms

$$(5) \quad \sigma_C : M[S] \twoheadrightarrow M[C] \hookrightarrow M[T].$$

We call σ_C a *canonical homomorphism* from $M[S]$ to $M[T]$ that factors through $M[C]$. It follows from [15] that each $\Lambda_{\bar{r}}$ -module homomorphism from $M[S]$ to $M[T]$ can be written uniquely as a \mathbb{k} -linear combination of canonical $\Lambda_{\bar{r}}$ -module homomorphisms as in (5). In particular, if $M[S] = M[T]$ then the canonical endomorphisms generate $\text{End}_{\Lambda_{\bar{r}}}(M[S])$.

3.4. Projective Indecomposable $\Lambda_{\bar{r}}$ -modules and modules whose endomorphism ring is isomorphic to \mathbb{k} . For all $i \in \{0, 1, 2\} \bmod 3$ vertex of Q , the radical series of the projective indecomposable

$\Lambda_{\bar{r}}$ -module P_i can be described as in the following figure.

$$(6) \quad \begin{array}{c} \begin{array}{ccc} & M[\mathbb{1}_i] & \\ \tau_i \swarrow & & \searrow \zeta_i \\ M[\mathbb{1}_{i+1}] & & M[\mathbb{1}_i] \\ \tau_{i+1} \downarrow & & \vdots \\ P_i = M[\mathbb{1}_{i+2}] & & \vdots \zeta_i^{r_i-2} \\ \tau_{i+2} \downarrow & & \vdots \\ M[\mathbb{1}_i] & & M[\mathbb{1}_i] \\ \searrow \underline{\zeta}_i^{k-1} \quad \swarrow \zeta_i & & \\ & M[\mathbb{1}_i] & \end{array} \end{array}$$

The following result provides a classification of all $\Lambda_{\bar{r}}$ -modules whose endomorphism ring is isomorphic to \mathbb{k} .

Proposition 3.1. *Let $M[S]$ be a string $\Lambda_{\bar{r}}$ -module, where $\bar{r} = (r_0, r_1, r_2, k)$ and $r_0, r_1, r_2 \geq 2$, $k \geq 1$. Then $M[S]$ has endomorphism ring isomorphic to \mathbb{k} if and only if for some $i \in \{0, 1, 2\} \bmod 3$ the string representative S is equivalent either to $\mathbb{1}_i$, or to τ_i , or to $\tau_{i+1}\tau_i$.*

Proof. If S is equivalent either to one of the strings $\mathbb{1}_0, \mathbb{1}_1$, or to $\mathbb{1}_2$, then it follows from Schur's Lemma that $\text{End}_{\Lambda}(M[S]) \cong \mathbb{k}$. If S is either equivalent to one of the strings $\tau_0, \tau_1, \tau_2, \tau_1\tau_0, \tau_2\tau_1$, or to $\tau_0\tau_2$ then the only canonical endomorphism in $\text{End}_{\Lambda_{\bar{r}}}(M[S])$ is the identity homomorphism, which implies that $\text{End}_{\Lambda_{\bar{r}}}(M[S])$ is one-dimensional over \mathbb{k} . Next assume that $M[S]$ is a string $\Lambda_{\bar{r}}$ -module with endomorphism ring isomorphic to \mathbb{k} . Let denote by n the length of S . If $n = 0$ then S is equivalent either to $\mathbb{1}_0$, or to $\mathbb{1}_1$, or to $\mathbb{1}_2$. If $n = 1$ then S is equivalent to an arrow. By hypothesis, S is equivalent neither to ζ_0 , nor to ζ_1 , nor to ζ_2 , for otherwise $\dim_{\mathbb{k}} \text{End}_{\Lambda_{\bar{r}}}(M[S]) \geq 2$. This implies that S is equivalent either to τ_0 , or to τ_1 , or to τ_2 . For the remainder of the proof, assume that $n \geq 2$ and let m be maximal such that the string representative S contains a substring equivalent to ζ_i^{-m} for some $i \in \{0, 1, 2\} \bmod 3$, and put $m = 0$ provided that S does not contain as substring any of the strings $\zeta_0, \zeta_1, \zeta_2$ or any of their formal inverses. If $m > 0$ then there exist suitable strings D and D' such that $S \sim D\zeta_i^{-m}D'$. It follows from the maximality of m that the string ζ_i^{-m} starts in a deep and ends on a peak. Therefore, there exists a non-trivial canonical endomorphism of $M[S]$ factoring through $M[\mathbb{1}_i]$ implying that $\dim_{\mathbb{k}} \text{End}_{\Lambda_{\bar{r}}}(M[S]) \geq 2$, which contradicts our hypothesis. Thus $m = 0$, implying that S does not contain as substrings the arrows ζ_0, ζ_1 , or ζ_2 or any of their formal inverses. Thus, there exist $i \in \{0, 1, 2\} \bmod 3$ and an integer $l \in \{0, \dots, k-1\}$ such that either $S \sim \underline{\zeta}_i^l \tau_{i+2}$ or $S \sim \underline{\zeta}_i^l \tau_{i+2} \tau_{i+1}$. If $l = 0$ then S is equivalent either to $\tau_1\tau_0$, or to $\tau_2\tau_1$, or to $\tau_0\tau_2$. Assume then that $l > 0$. If $S \sim \underline{\zeta}_i^l \tau_{i+2}$ (respectively, $S \sim \underline{\zeta}_i^l \tau_{i+2} \tau_{i+1}$) then there exists a non-trivial canonical endomorphism of $M[S]$ factoring through $M[\tau_{i+2}]$ (respectively, through $M[\tau_{i+2}\tau_{i+1}]$) implying that $\dim_{\mathbb{k}} \text{End}_{\Lambda_{\bar{r}}}(M[S]) \geq 2$, contradicting again our hypothesis. This finishes the proof of Proposition 3.1. \square

4. COMPONENTS OF $\Gamma_s(\Lambda_{\bar{r}})$ OF TYPE $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$ CONTAINING A MODULE WHOSE ENDOMORPHISM RING IS ISOMORPHIC TO \mathbb{k} AND 3-TUBES

For all $i \in \{0, 1, 2\} \bmod 3$ we define:

$$(7) \quad \underline{a}_i = \tau_i \zeta_i^{-r_i+1} \quad \underline{b}_i = \underline{\zeta}_{i+2}^{k-1} \tau_{i+1} \tau_i \zeta_i^{-1}$$

4.1. Components of $\Gamma_s(\Lambda_{\bar{r}})$ of type $\mathbb{Z}\mathbb{A}_{\infty}^{\infty}$ containing a module whose endomorphism ring is isomorphic to \mathbb{k} .

Proposition 4.1. *For $i \in \{0, 1, 2\} \bmod 3$, let \mathfrak{A}_i be the component of the stable Auslander-Reiten quiver of $\Lambda_{\bar{r}}$ containing the simple $\Lambda_{\bar{r}}$ -module $M[\mathbb{1}_i]$, where $\bar{r} = (r_0, r_1, r_2, k)$ and $r_0, r_1, r_2 \geq 2$, $k \geq 1$. Define*

$$(\mathbb{1}_i)_h = \underline{a}_{i+2} \quad \text{and} \quad (\mathbb{1}_i)_{hh} = \underline{a}_{i+2}\underline{a}_{i+1}.$$

The component \mathfrak{A}_i is Ω -stable if and only if for $r_i = 2$. If $k = 1$ then the module $M[\tau_{i+1}]$ lies in $\Omega(\mathfrak{A}_i)$. The modules in $\mathfrak{A}_i \cup \Omega(\mathfrak{A}_i)$ whose stable endomorphism rings are isomorphic to \mathbb{k} are precisely the modules in Ω -orbits of the modules $U_0 = M[\mathbb{1}_i]$, $U_1 = M[(\mathbb{1}_i)_h]$ and $U_2 = M[(\mathbb{1}_i)_{hh}]$. Their universal deformation rings are

$$R(\Lambda_{\bar{r}}, U_0) \cong \mathbb{k}[[t]]/(t^{r_i}), \quad R(\Lambda_{\bar{r}}, U_1) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, U_2) \cong \mathbb{k}[[t]].$$

Proof. Let $i \in \{0, 1, 2\} \bmod 3$ be fixed. Using hooks and co-hooks (see §3.2), we see that all $\Lambda_{\bar{r}}$ -modules in $\mathfrak{A}_i \cup \Omega(\mathfrak{A}_i)$ lie in the Ω -orbit of either

$$\begin{aligned} A_{q,0} &= M[(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q], \text{ or} \\ A_{q,1} &= M[(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q\underline{a}_{i+2}], \text{ or} \\ A_{q,2} &= M[(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q\underline{a}_{i+2}\underline{a}_{i+1}], \text{ or} \\ B_{q,0} &= M[(\underline{b}_{i+1}\underline{b}_{i+2}\underline{b}_i)^q], \text{ or} \\ B_{q,1} &= M[\underline{b}_i(\underline{b}_{i+1}\underline{b}_{i+2}\underline{b}_i)^q], \text{ or} \\ B_{q,2} &= M[\underline{b}_{i+2}\underline{b}_i(\underline{b}_{i+1}\underline{b}_{i+2}\underline{b}_i)^q] \end{aligned}$$

for some $q \geq 0$. Note for example that $A_{0,0} = M[\mathbb{1}_i] = B_{0,0}$, $A_{0,1} = M[(\mathbb{1}_i)_h]$, $A_{0,2} = M[(\mathbb{1}_i)_{hh}]$, $B_{0,1} = M[h(\mathbb{1}_i)]$ and $B_{0,2} = M[h_h(\mathbb{1}_i)]$. Since $\Omega M[\mathbb{1}_i] = M[\zeta_i^{-r_i+1}\underline{a}_i^{k-1}\tau_{i+2}\tau_{i+1}]$ then $\mathfrak{A}_i = \Omega(\mathfrak{A}_i)$ if and only if $r_i = 2$; and that if $k = 1$ then $\Omega(M[\mathbb{1}_i]) = M[c(\tau_{i+1})]$.

Using §3.3 and the description of the projective indecomposable $\Lambda_{\bar{r}}$ -module P_i in (6), it is straight forward to show that the stable endomorphism ring of $A_{0,j}$ is isomorphic to \mathbb{k} for $j \in \{0, 1, 2\}$ and that $\text{Ext}_{\Lambda_{\bar{r}}}^1(A_{0,j}, A_{0,j})$ is isomorphic to \mathbb{k} for $j \in \{0, 2\}$ and zero for $j = 1$. On the other hand, for $q \geq 1$ and for $j \in \{0, 1, 2\}$, the $\Lambda_{\bar{r}}$ -module $A_{q,j}$ has a non-zero endomorphism which factors through $M[\mathbb{1}_i]$ and which does not factor through a projective $\Lambda_{\bar{r}}$ -module. Assume that $r_i = 2$. Since in this case \mathfrak{A}_i is Ω -stable, then for all $j \in \{0, 1, 2\} \bmod 3$ and for all $q \geq 0$, the $\Lambda_{\bar{r}}$ -module $B_{q,j}$ lies in the Ω -orbit of $A_{q',j'}$ for some $j' \in \{0, 1, 2\}$ and $q' \geq 0$. In particular, $B_{0,1} = \Omega^{-1}A_{0,0}$, $B_{0,2} = \Omega^{-1}A_{0,1}$ and $B_{1,0} = \Omega^{-1}A_{0,2}$. If $r_i \geq 3$ then each of the modules $B_{0,1}$, $B_{0,2}$ and $B_{q,j}$ with $j \in \{0, 1, 2\}$ and $q \geq 1$ have a non-zero endomorphism factoring through $M[\mathbb{1}_i]$ and which does not factor through a projective $\Lambda_{\bar{r}}$ -module. Therefore, for all $r_i \geq 2$, the modules in $\mathfrak{A}_i \cup \Omega(\mathfrak{A}_i)$ whose stable endomorphism rings are isomorphic to \mathbb{k} are precisely the modules in Ω -orbits of the modules $A_{0,0}$, $A_{0,1}$ and $A_{0,2}$.

Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(A_{0,1}, A_{0,1}) = 0$, it follows that $R(\Lambda_{\bar{r}}, A_{0,1}) \cong \mathbb{k}$. Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(A_{0,j}, A_{0,j})$ is isomorphic to \mathbb{k} for $j \in \{0, 2\}$, it follows that $R(\Lambda_{\bar{r}}, A_{0,j})$ is a quotient of $\mathbb{k}[[t]]$ for $j \in \{0, 2\}$.

Let consider the $\Lambda_{\bar{r}}$ -module $A_{0,0} = M[\mathbb{1}_i]$.

Claim 4.2. The universal deformation ring $R(\Lambda_{\bar{r}}, A_{0,0})$ of $A_{0,0}$ is isomorphic to $\mathbb{k}[[t]]/(t^{r_i})$.

Proof of Claim. For all $l \in \{0, \dots, r_i - 1\}$ let $S_l = \zeta_i^{-l}$. Then for all $l \in \{1, \dots, r_i - 1\}$ there exists a non-trivial canonical endomorphism σ_l of the $\Lambda_{\bar{r}}$ -module $M[S_l]$ which factors through $M[S_{l-1}]$, namely

$$(8) \quad \sigma_l : M[S_l] \rightarrow M[S_{l-1}] \hookrightarrow M[S_l].$$

Observe that the kernel of σ_l and the image of σ_l^{l-1} are isomorphic to $A_{0,0}$, and that $\sigma_l^l = 0$. Thus, for all $l \in \{0, \dots, r_i - 1\}$, the $\Lambda_{\bar{r}}$ -module $M[S_l]$ is naturally a $\mathbb{k}[[t]]/(t^{l+1}) \otimes_{\mathbb{k}} \Lambda_{\bar{r}}$ -module where the action of t over $m \in M[S_l]$ is given as $t \cdot m = \sigma_l(m)$. In particular, $tM[S_l] \cong M[S_{l-1}]$ for all $l \in \{1, \dots, r_i - 1\}$.

Let $l \in \{1, \dots, r_i - 1\}$ be fixed and let $\{\bar{b}_1\}$ be a \mathbb{k} -basis of $A_{0,0}$. Using the isomorphism $M[S_l]/tM[S_l] \cong A_{0,0}$, we can lift \bar{b}_1 to an element $b_1 \in M[S_l]$. It follows that $\{b_1\}$ is linearly independent over \mathbb{k} and that $\{t^a b_1 : 0 \leq a \leq l\}$ is a \mathbb{k} -basis of $tM[S_l] \cong M[S_{l-1}]$. Therefore, $\{b_1\}$ is a $\mathbb{k}[[t]]/(t^{l+1})$ -basis of $M[S_l]$, which means that $M[S_l]$ is free over $\mathbb{k}[[t]]/(t^{l+1})$. Moreover, $M[S_l]$ lies in a short exact sequences of $\Lambda_{\bar{r}}$ -modules

$$0 \rightarrow tM[S_l] \rightarrow M[S_l] \rightarrow \mathbb{k} \otimes_{\mathbb{k}[[t]]/(t^{l+1})} M[S_l] \rightarrow 0.$$

Consequently, there exists an isomorphism of $\Lambda_{\bar{r}}$ -modules $\phi_l : \mathbb{k} \otimes_{\mathbb{k}[[t]]/(t^{l+1})} M[S_l] \rightarrow A_{0,0}$, which implies that $(M[S_l], \phi_l)$ is a lift of $A_{0,0}$ over $\mathbb{k}[[t]]/(t^{l+1})$. Consider the lift $(M[S_{r_i-1}], \phi_{r_i-1})$ of $A_{0,0}$ over $\mathbb{k}[[t]]/(t^{r_i})$. Since $\text{End}_{\Lambda_{\bar{r}}}(A_{0,0}) \cong \mathbb{k}$ then by Theorem 2.3(i), there exists a unique morphism $\alpha : R(\Lambda_{\bar{r}}, A_{0,0}) \rightarrow \mathbb{k}[[t]]/(t^{r_i})$ in $\hat{\mathcal{C}}$ such that $M[S_{r_i-1}] \cong \mathbb{k}[[t]]/(t^{r_i}) \otimes_{R(\Lambda_{\bar{r}}, A_{0,0}), \alpha} U(\Lambda_{\bar{r}}, A_{0,0})$, where $R(\Lambda_{\bar{r}}, A_{0,0})$ and $U(\Lambda_{\bar{r}}, A_{0,0})$ are respectively the universal deformation ring and the universal deformation of the $\Lambda_{\bar{r}}$ -module $A_{0,0}$. Since $(M[S_1], \phi_1)$ is not the trivial lift of $A_{0,0}$ over $\mathbb{k}[[t]]/(t^2)$, it follows that there exists a unique surjective morphism $\alpha' : R(\Lambda_{\bar{r}}, A_{0,0}) \rightarrow \mathbb{k}[[t]]/(t^2)$ in $\hat{\mathcal{C}}$ such that $M[S_1] \cong \mathbb{k}[[t]]/(t^2) \otimes_{R(\Lambda_{\bar{r}}, A_{0,0}), \alpha'} U(\Lambda_{\bar{r}}, A_{0,0})$. By considering the natural projection $\pi_{r_i,2} : \mathbb{k}[[t]]/(t^{r_i}) \rightarrow \mathbb{k}[[t]]/(t^2)$ and the lift $(U', \phi_{U'})$ of $A_{0,0}$ over $\mathbb{k}[[t]]/(t^2)$ corresponding to the morphism $\pi_{r_i,2} \circ \alpha$, we obtain

$$\begin{aligned} U' &\cong \mathbb{k}[[t]]/(t^2) \otimes_{R(\Lambda_{\bar{r}}, A_{0,0}), \pi_{r_i,2} \circ \alpha} U(\Lambda_{\bar{r}}, A_{0,0}) \\ &\cong \mathbb{k}[[t]]/(t^2) \otimes_{\mathbb{k}[[t]]/(t^{r_i}), \pi_{r_i,2}} (\mathbb{k}[[t]]/(t^{r_i}) \otimes_{R(\Lambda_{\bar{r}}, A_{0,0}), \alpha} U(\Lambda_{\bar{r}}, A_{0,0})) \\ &\cong \mathbb{k}[[t]]/(t^2) \otimes_{\mathbb{k}[[t]]/(t^{r_i}), \pi_{r_i,2}} M[S_{r_i-1}] \\ &\cong M[S_{r_i-1}]/t^2 M[S_{r_i-1}] \cong M[S_1]. \end{aligned}$$

It follows from Remark 2.1 that $[U', \phi_{U'}] = [M[S_1], \phi_1]$ in $\hat{F}_{A_{0,0}}(\mathbb{k}[[t]]/(t^2))$. The uniqueness of α' implies $\alpha' = \pi_{r_i,2} \circ \alpha$. Since α' is surjective, it follows that α is also surjective. We want to prove that α is an isomorphism. Suppose this is false. Then there exists a surjective \mathbb{k} -algebra homomorphism $\alpha_0 : R(\Lambda_{\bar{r}}, A_{0,0}) \rightarrow \mathbb{k}[[t]]/(t^{r_i+1})$ in $\hat{\mathcal{C}}$ such that $\pi_{r_i+1, r_i} \circ \alpha_0 = \alpha$, where $\pi_{r_i+1, r_i} : \mathbb{k}[[t]]/(t^{r_i+1}) \rightarrow \mathbb{k}[[t]]/(t^{r_i})$ is the natural projection. Let M_0 be a $\mathbb{k}[[t]]/(t^{r_i+1}) \otimes_{\mathbb{k}} \Lambda_{\bar{r}}$ -module which defines a lift of $A_{0,0}$ over $\mathbb{k}[[t]]/(t^{r_i+1})$ corresponding to α_0 . Since the kernel of π_{r_i+1, r_i} is $(t^{r_i})/(t^{r_i+1})$, then $M_0/t^{r_i} M_0 \cong M[S_{r_i-1}]$. Consider the $\mathbb{k}[[t]]/(t^{r_i+1}) \otimes_{\mathbb{k}} \Lambda_{\bar{r}}$ -module homomorphism $g : M_0 \rightarrow t^{r_i} M_0$ defined by $g(m) = t^{r_i} m$ for all $m \in M_0$. Since M_0 is free over $\mathbb{k}[[t]]/(t^{r_i+1})$, it follows that the kernel of g is isomorphic to $t M_0$. Since g is a surjection, it follows that $M_0/t M_0 \cong t^{r_i} M_0$, which implies that $t^{r_i} M_0 \cong A_{0,0}$. Hence, there exists a non-split short exact sequence of $\mathbb{k}[[t]]/(t^{r_i+1}) \otimes_{\mathbb{k}} \Lambda_{\bar{r}}$ -modules

$$(9) \quad 0 \rightarrow A_{0,0} \rightarrow M_0 \rightarrow M[S_{r_i-1}] \rightarrow 0.$$

Since $\Omega M[S_{r_i-1}] = \Omega M[\zeta_i^{-r_i+1}] = M[\zeta_i^{k-1} \tau_{i+2} \tau_{i+1}]$, then

$$\text{Ext}_{\Lambda_{\bar{r}}}^1(M[S_{r_i-1}], A_{0,0}) = \underline{\text{Hom}}_{\Lambda_{\bar{r}}}(\Omega M[S_{r_i-1}], A_{0,0}) = 0.$$

It follows that the sequence (9) splits as a sequence of $\Lambda_{\bar{r}}$ -modules. Hence $M_0 = A_{0,0} \oplus M[S_{r_i-1}]$ as $\Lambda_{\bar{r}}$ -modules. Identifying the elements of M_0 as (a, m) with $a \in A_{0,0}$ and $m \in M[S_{r_i-1}]$ we see that the t acts on $(a, m) \in M_0$ as $t \cdot (a, m) = (\mu(m), \sigma_{r_i-1}(m))$, where $\mu : M[S_{r_i-1}] \rightarrow A_{0,0}$ is a surjective $\Lambda_{\bar{r}}$ -module homomorphism and σ_{r_i-1} is as in (8). Since the canonical homomorphism $\epsilon : M[S_{r_i-1}] \twoheadrightarrow M[\mathbb{1}_i] \rightarrow M[\mathbb{1}_i]$ generates $\text{Hom}_{\Lambda_{\bar{r}}}(M[S_{r_i-1}], A_{0,0})$, then there exists $c \in \mathbb{k}^*$ such that $\mu = c\epsilon$, which implies that the kernel of μ is $t M[S_{r_i-1}]$. Therefore $t^{r_i}(a, m) = (\mu(t^{r_i-1} m), \sigma_{r_i-1}^{r_i}(m)) = (0, 0)$ for all $a \in A_{0,0}$ and $m \in M[S_{r_i-1}]$, which contradicts the fact that $t^{r_i} M_0 \cong A_{0,0}$. Thus $\alpha : R(\Lambda_{\bar{r}}, A_{0,0}) \rightarrow \mathbb{k}[[t]]/(t^{r_i})$ is an isomorphism and $R(\Lambda_{\bar{r}}, A_{0,0}) \cong \mathbb{k}[[t]]/(t^{r_i})$. This finishes the proof of Claim 4.2.

Next consider the string $\Lambda_{\bar{r}}$ -module $A_{0,2} = M[(\mathbb{1}_i)_{hh}]$.

Claim 4.3. The universal deformation ring $R(\Lambda_{\bar{r}}, A_{0,2})$ of $A_{0,2}$ is isomorphic to $\mathbb{k}[[t]]$.

Proof of Claim. Let $T_0 = (\mathbb{1}_i)_{hh}$ and for all $l \geq 1$, let $T_l = T_{l-1} \tau_i(\mathbb{1}_i)_{hh}$. Thus, for all $l \geq 1$ and by using similar arguments as those in the proof of Claim 4.2, we get lifts $(M[T_l], \varphi_l)$ of $A_{0,2}$ over $\mathbb{k}[[t]]/(t^{l+1})$, where for each $l \geq 1$, t acts on $m \in M[T_l]$ as $t \cdot m = \delta_l(m)$, where δ_l is the non-trivial canonical endomorphism of $M[T_l]$ that factors through $M[T_{l-1}]$, namely

$$(10) \quad \delta_l : M[T_l] \twoheadrightarrow M[T_{l-1}] \hookrightarrow M[T_l].$$

Note that for all $l \geq 1$, we have natural projections $\pi_{l,l-1} : M[T_l] \rightarrow M[T_{l-1}]$. Let $N_0 = \varprojlim M[T_l]$ and let t act on N_0 as $\varprojlim \pi_{l,l-1}$. In particular, $\mathbb{k} \otimes_{\mathbb{k}[[t]]} N_0 \cong N_0/t N_0 \cong A_{0,2}$, which implies that there exists an isomorphism of $\Lambda_{\bar{r}}$ -modules $\varphi_0 : \mathbb{k} \otimes_{\mathbb{k}[[t]]} N_0 \rightarrow A_{0,2}$. Let $n = \dim_{\mathbb{k}} A_{0,2}$ and let $\{\bar{B}_j\}_{1 \leq j \leq n}$ be a \mathbb{k} -basis of

N_0/tN_0 . For all $1 \leq j \leq n$, we are able to lift these elements \bar{B}_j in N_0/tN_0 to elements B_j of N_0 such that $\{B_j\}_{1 \leq j \leq n}$ is a generating set of the $\mathbb{k}[[t]] \otimes_{\mathbb{k}} \Lambda_{\bar{r}}$ -module N_0 . It follows that $\{B_j\}_{1 \leq j \leq n}$ is a $\mathbb{k}[[t]]$ -basis of N_0 , which implies that N_0 is free over $\mathbb{k}[[t]]$. Therefore, (N_0, φ_0) is a lift of $A_{0,2}$ over $\mathbb{k}[[t]]$ and there exists a unique \mathbb{k} -algebra homomorphism $\beta : R(\Lambda_{\bar{r}}, A_{0,2}) \rightarrow \mathbb{k}[[t]]$ in $\hat{\mathcal{C}}$ corresponding to the deformation defined by (N_0, φ_0) , where $R(\Lambda_{\bar{r}}, A_{0,2})$ is the universal deformation ring of $A_{0,2}$. Since $N_0/t^2N_0 \cong M[T_1]$ as $\Lambda_{\bar{r}}$ -modules, we can see as in the proof of Claim 4.2 that since N_0/t^2N_0 defines a non-trivial lift of $A_{0,2}$ over $\mathbb{k}[[t]]/(t^2)$, then β is a surjection. Since $R(\Lambda_{\bar{r}}, A_{0,2})$ is a quotient of $\mathbb{k}[[t]]$, it follows that β is an isomorphism. Hence $R(\Lambda_{\bar{r}}, A_{0,2}) \cong \mathbb{k}[[t]]$. This finishes the proof of Claim 4.3, which finishes the proof of Proposition 4.1. \square

Proposition 4.4. *For $i \in \{0, 1, 2\} \bmod 3$, let \mathfrak{B}_i be the component of $\Gamma_s(\Lambda_{\bar{r}})$ containing the $\Lambda_{\bar{r}}$ -module $M[\tau_i]$, where $\bar{r} = (r_0, r_1, r_2, k)$ and $r_0, r_1, r_2 \geq 3$, $k \geq 1$. Define*

$$(\tau_i)_h = \tau_i \underline{a}_{i+2} \quad \text{and} \quad {}_h(\tau_i) = \underline{b}_{i+1} \tau_i.$$

If $k = 1$ then $\mathfrak{B}_i = \Omega(\mathfrak{A}_{i+2})$, where \mathfrak{A}_{i+2} is as in Proposition 4.1. Thus, $\mathfrak{B}_i = \Omega(\mathfrak{B}_i)$ if and only if $k = 1$ and $r_{i+2} = 2$. The modules in $\mathfrak{B}_i \cup \Omega(\mathfrak{B}_i)$ whose stable endomorphism rings are isomorphic to \mathbb{k} are precisely the modules in the Ω -orbits of the modules $V_0 = M[\tau_i]$, $V_1 = M[(\tau_i)_h]$ and $V_{-1} = M[{}_h(\tau_i)]$. If $k = 1$ then the universal deformation rings are

$$R(\Lambda_{\bar{r}}, V_0) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, V_1) \cong \mathbb{k}[[t]]/(t^{r_{i+2}}), \quad R(\Lambda_{\bar{r}}, V_{-1}) \cong \mathbb{k}[[t]].$$

If $k \geq 2$ then the universal deformation rings are

$$R(\Lambda_{\bar{r}}, V_0) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, V_1) \cong \mathbb{k}[[t]], \quad R(\Lambda_{\bar{r}}, V_{-1}) \cong \mathbb{k}[[t]].$$

Proof. Let $i \in \{0, 1, 2\} \bmod 3$ be fixed. Using hooks and co-hooks (see §3.2) we see that all $\Lambda_{\bar{r}}$ -modules in \mathfrak{B}_i lie in the Ω -orbit of either

$$\begin{aligned} C_{q,0} &= M[\tau_i(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q], \text{ or} \\ C_{q,1} &= M[\tau_i(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q\underline{a}_{i+2}], \text{ or} \\ C_{q,2} &= M[\tau_i(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q\underline{a}_{i+2}\underline{a}_{i+1}], \text{ or} \\ D_{q,0} &= M[(\underline{b}_{i+2}\underline{b}_i\underline{b}_{i+1})^q\tau_i], \text{ or} \\ D_{q,1} &= M[\underline{b}_{i+1}(\underline{b}_{i+2}\underline{b}_i\underline{b}_{i+1})^q\tau_i], \text{ or} \\ D_{q,2} &= M[\underline{b}_i\underline{b}_{i+1}(\underline{b}_{i+2}\underline{b}_i\underline{b}_{i+1})^q\tau_i] \end{aligned}$$

for some $q \geq 0$.

Note that $C_{0,0} = M[\tau_i] = D_{0,0}$, $C_{0,1} = M[(\tau_i)_h]$, $C_{0,2} = M[(\tau_i)_{hh}]$, $D_{0,1} = M[{}_h(\tau_i)]$ and $D_{0,2} = M[{}_h{}_h(\tau_i)]$. By Proposition 4.1, it follows that if $k = 1$ then $M[\tau_i]$ lies in $\Omega(\mathfrak{A}_{i+2})$, which implies that \mathfrak{B}_i is Ω -stable if and only if $r_{i+2} = 2$ and $k = 1$.

Using §3.3 and the description of the projective indecomposable $\Lambda_{\bar{r}}$ -module P_i in (6), it is straight forward to show that the stable endomorphism rings of $C_{0,j}$ and $D_{0,j}$ are isomorphic to \mathbb{k} for $j \in \{0, 1\}$, that $\text{Ext}_{\Lambda_{\bar{r}}}^1(C_{0,1}, C_{0,1})$ and $\text{Ext}_{\Lambda_{\bar{r}}}^1(D_{0,1}, D_{0,1})$ are isomorphic to \mathbb{k} , and that $\text{Ext}_{\Lambda_{\bar{r}}}^1(C_{0,0}, C_{0,0}) = 0$. Moreover, $D_{0,2}$ and $D_{q,j}$ with $q \geq 1$ and $j \in \{0, 1, 2\}$ have a non-zero canonical endomorphism factoring through $M[\tau_i]$ that does not factor through a projective $\Lambda_{\bar{r}}$ -module. If $k \geq 2$ or $r_{i+2} \geq 3$ then $C_{0,2}$ and $C_{q,j}$ with $q \geq 1$ and $j \in \{0, 1, 2\}$ have a non-zero canonical endomorphism which factors through $M[\mathbb{1}_{i+1}]$ and which does not factor through a projective $\Lambda_{\bar{r}}$ -module. If $k = 1$ and $r_{i+2} = 2$ then $C_{0,2} = \Omega^{-1}C_{0,1}$, $C_{1,0} = \Omega^{-3}C_{0,0}$, $C_{1,1} = \Omega^{-3}D_{0,1}$, and the modules $C_{1,2}$ and $C_{q,j}$ with $q \geq 2$, $j \in \{0, 1, 2\}$ have a non-trivial canonical endomorphism factoring through $M[\mathbb{1}_{i+1}]$ that does not factor through a projective $\Lambda_{\bar{r}}$ -module. Therefore, for all $r_0, r_1, r_2 \geq 2$ and $k \geq 1$, the modules in $\mathfrak{B}_i \cup \Omega(\mathfrak{B}_i)$ whose stable endomorphism rings are isomorphic to \mathbb{k} are precisely the modules in the Ω -orbits of $C_{0,0}$, $C_{0,1}$ and $D_{0,1}$.

Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(C_{0,0}, C_{0,0}) = 0$, it follows that $R(\Lambda_{\bar{r}}, C_{0,0}) \cong \mathbb{k}$. Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(C_{0,1}, C_{0,1})$ and $\text{Ext}_{\Lambda_{\bar{r}}}^1(D_{0,1}, D_{0,1})$ are both isomorphic to \mathbb{k} then $R(\Lambda_{\bar{r}}, C_{0,1})$ and $R(\Lambda_{\bar{r}}, D_{0,1})$ are quotients of $\mathbb{k}[[t]]$. Assume that $k = 1$. Then

by Theorem 2.3 and Proposition 4.1, it follows that

$$R(\Lambda_{\bar{r}}, C_{0,1}) \cong R(\Lambda_{\bar{r}}, \Omega^{-2}M[\mathbb{1}_{i+2}]) \cong \mathbb{k}[[t]]/(t^{r_{i+2}}),$$

and

$$R(\Lambda_{\bar{r}}, D_{0,1}) \cong R(\Lambda_{\bar{r}}, \Omega M[(\mathbb{1}_{i+2})_{hh}]) \cong \mathbb{k}[[t]].$$

Next assume that $k \geq 2$. Let $S_0 = (\tau_i)_h$, $T_0 = {}_h(\tau_i)$ and for all $l \geq 1$, let $S_l = S_{l-1}\tau_{i+1}(\tau_i)_h$ and $T_l = T_{l-1}\zeta_i^{-1}{}_h(\tau_i)$. Then by using similar arguments as in proof of Claim 4.3 within the proof of Proposition 4.1, we obtain that $R(\Lambda_{\bar{r}}, C_{0,1}) \cong \mathbb{k}[[t]] \cong R(\Lambda_{\bar{r}}, D_{0,1})$. This finishes the proof of Proposition 4.4. \square

Let \mathfrak{C}_i be the component of $\Gamma_s(\Lambda_{\bar{r}})$ containing the string module $M[\tau_{i+1}\tau_i]$ for some $i \in \{0, 1, 2\} \bmod 3$. Observe that if $k = 1$ then \mathfrak{C}_i is one of the 3-tubes, otherwise \mathfrak{C}_i is a component of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. In Proposition 4.6, we determine the universal deformation rings of modules whose stable endomorphism ring is isomorphic to \mathbb{k} lying in the 3-tubes (see Proposition 4.6). In the following result, we assume that $k \geq 2$.

Proposition 4.5. *For $i \in \{0, 1, 2\} \bmod 3$, let \mathfrak{C}_i be the component of $\Gamma_s(\Lambda_{\bar{r}})$ containing the $\Lambda_{\bar{r}}$ -module $M[\tau_{i+1}\tau_i]$, where $\bar{r} = (r_0, r_1, r_2, k)$ and $r_0, r_1, r_2 \geq 2$, $k \geq 2$. Define*

$${}_h(\tau_{i+1}\tau_i) = \underline{b}_{i+2}\tau_{i+1}\tau_i \quad \text{and} \quad {}_{hh}(\tau_{i+1}\tau_i) = \underline{b}_{i+1}\underline{b}_{i+2}\tau_{i+1}\tau_i.$$

The component \mathfrak{C}_i is Ω -stable if and only if $k = 2$. The modules in \mathfrak{C}_i whose stable endomorphism ring is isomorphic to \mathbb{k} are precisely the modules in the Ω -orbits of the modules $W_0 = M[\tau_{i+1}\tau_i]$, $W_{-1} = M[{}_h(\tau_{i+1}\tau_i)]$ and $W_{-2} = M[{}_{hh}(\tau_{i+1}\tau_i)]$. Their universal deformation rings are

$$R(\Lambda_{\bar{r}}, W_0) \cong \mathbb{k}[[t]]/(t^k), \quad R(\Lambda_{\bar{r}}, W_{-1}) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, W_{-2}) \cong \mathbb{k}[[t]].$$

Proof. Let $i \in \{0, 1, 2\} \bmod 3$ be fixed. Using hooks and co-hooks (see §3.2) we see that all $\Lambda_{\bar{r}}$ -modules in \mathfrak{C}_i lie in the Ω -orbit of either

$$\begin{aligned} E_{q,0} &= M[\tau_{i+1}\tau_i(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q], \text{ or} \\ E_{q,1} &= M[\tau_{i+1}\tau_i(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q\underline{a}_{i+2}], \text{ or} \\ E_{q,2} &= M[\tau_{i+1}\tau_i(\underline{a}_{i+2}\underline{a}_{i+1}\underline{a}_i)^q\underline{a}_{i+2}\underline{a}_{i+1}], \text{ or} \\ F_{q,0} &= M[(\underline{b}_i\underline{b}_{i+1}\underline{b}_{i+2})^q\tau_{i+1}\tau_i], \text{ or} \\ F_{q,1} &= M[\underline{b}_{i+2}(\underline{b}_i\underline{b}_{i+1}\underline{b}_{i+2})^q\tau_{i+1}\tau_i], \text{ or} \\ F_{q,2} &= M[\underline{b}_{i+1}\underline{b}_{i+2}(\underline{b}_i\underline{b}_{i+1}\underline{b}_{i+2})^q\tau_{i+1}\tau_i] \end{aligned}$$

for some $q \geq 0$. Note that $E_{0,0} = M[\tau_{i+1}\tau_i] = F_{0,0}$, $E_{0,1} = M[(\tau_{i+1}\tau_i)_h]$, $E_{0,2} = M[(\tau_{i+1}\tau_i)_{hh}]$, $F_{0,1} = M[{}_h(\tau_{i+1}\tau_i)]$ and $F_{0,2} = M[{}_{hh}(\tau_{i+1}\tau_i)]$. Since $\Omega F_{0,0} = M[\underline{c}(\tau_{i+1}\tau_i)\underline{c}^{k-2}]$, then \mathfrak{C}_i is Ω -stable if and only if $k = 2$.

By using §3.3 and the description of the projective indecomposable $\Lambda_{\bar{r}}$ -module P_i in (6), it is straight forward to show that for all $j \in \{0, 1, 2\}$, the stable endomorphism ring of $F_{0,j}$ is isomorphic to \mathbb{k} and for $q \geq 1$, the module $F_{q,j}$ has a non-trivial canonical endomorphism which factors through $M[\tau_{i+1}\tau_i]$ and which does not factor through a projective $\Lambda_{\bar{r}}$ -module.

Assume first that $k = 2$. Since in this case \mathfrak{C}_i is Ω -stable, then for all $j \in \{0, 1, 2\} \bmod 3$ and for all $q \geq 0$ the $\Lambda_{\bar{r}}$ -module $E_{q,j}$ lies in the Ω -orbit of $F_{q',j'}$ for some $j' \in \{0, 1, 2\}$ and $q' \geq 0$. In particular, $E_{0,1} = \Omega^{-1}F_{0,0}$, $E_{0,2} = \Omega^{-1}F_{0,1}$ and $E_{1,0} = \Omega^{-1}F_{0,2}$. Next assume that $k \geq 3$. Then for all $q \geq 0$ and $j \in \{0, 1, 2\}$ the module $E_{q,j}$ has a non-trivial canonical endomorphism, which factors through $M[\mathbb{1}_{i+2}]$ and which does not factor through a projective $\Lambda_{\bar{r}}$ -module. Therefore for all $k \geq 2$, the modules in \mathfrak{C}_i whose stable endomorphism ring is isomorphic to \mathbb{k} are precisely the modules in the Ω -orbits of the modules $F_{0,0}$, $F_{0,1}$ and $F_{0,2}$.

Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(F_{0,1}, F_{0,1}) = 0$, it follows that $R(\Lambda_{\bar{r}}, F_{0,1}) \cong \mathbb{k}$. Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(F_{0,0}, F_{0,0})$ and $\text{Ext}_{\Lambda_{\bar{r}}}^1(F_{0,2}, F_{0,2})$ are isomorphic to \mathbb{k} then $R(\Lambda_{\bar{r}}, F_{0,0})$ and $R(\Lambda_{\bar{r}}, F_{0,2})$ are quotients of $\mathbb{k}[[t]]$. Let $T_0 = {}_{hh}(\tau_{i+1}\tau_i)$ and for all $0 \leq j \leq k-1$ and $l \geq 1$, let $S_j = \underline{c}_{i+2}^j\tau_{i+1}\tau_i$ and $T_l = {}_{hh}(\tau_{i+1}\tau_i)\zeta_i^{-1}T_{l-1}$. By using similar arguments as those in the proof of Proposition 4.1, we obtain that $R(\Lambda_{\bar{r}}, F_{0,0}) \cong \mathbb{k}[[t]]/(t^k)$ and $R(\Lambda_{\bar{r}}, F_{0,2}) \cong \mathbb{k}[[t]]$. This finishes the proof of Proposition 4.5. \square

4.2. 3-tubes.

Proposition 4.6. *Let \mathfrak{T}_1 and \mathfrak{T}_2 be the two 3-tubes of $\Gamma_s(\Lambda_{\bar{r}})$, with $\bar{r} = (r_0, r_1, r_2, k)$ and $r_0, r_1, r_2 \geq 2$ and $k \geq 1$. Then $\Omega(\mathfrak{T}_1) = \mathfrak{T}_2$. Let $T = \zeta_0^{-r_0+1}$ and define*

$$T_h = \zeta_0^{-r_0+1} \tau_2 \zeta_2^{-r_2+1} \quad \text{and} \quad T_{hh} = \zeta_0^{-r_0+1} \tau_2 \zeta_2^{-r_2+1} \tau_1 \zeta_1^{-r_1+1}.$$

The modules in $\mathfrak{T}_1 \cup \mathfrak{T}_2$ whose stable endomorphism rings are isomorphic to \mathbb{k} are precisely the modules in the Ω -orbit of $X_0 = M[T]$, $X_1 = M[T_h]$ and $X_2 = M[T_{hh}]$. Their universal deformation rings are

$$R(\Lambda_{\bar{r}}, X_0) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, X_1) \cong \mathbb{k}, \quad R(\Lambda_{\bar{r}}, X_2) \cong \mathbb{k}[[t]]$$

Proof. Using the description of the projective indecomposable $\Lambda_{\bar{r}}$ -modules in (6), we see that $\Omega(\mathfrak{T}_1) = \mathfrak{T}_2$. Using §3.3 and the description of the projective indecomposable $\Lambda_{\bar{r}}$ -module P_i in (6), it is straightforward to show that the only $\Lambda_{\bar{r}}$ -modules in $\mathfrak{T}_1 \cup \mathfrak{T}_2$ whose stable endomorphism rings are isomorphic to \mathbb{k} lie in the Ω -orbit of either $X_0 = M[T]$, $X_1 = M[T_h]$ or $X_2 = M[T_{hh}]$. Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(X_j, X_j) = 0$ for $j \in \{0, 1\}$, we have that $R(\Lambda_{\bar{r}}, X_j) \cong \mathbb{k}$ for $j \in \{0, 1\}$. Since $\text{Ext}_{\Lambda_{\bar{r}}}^1(X_2, X_2)$ is isomorphic to \mathbb{k} , it follows that $R(\Lambda_{\bar{r}}, X_2)$ is a quotient of $\mathbb{k}[[t]]$. Let $S_0 = T_{hh}$ and for all $l \geq 1$, let $S_l = S_{l-1} \tau_i T_{hh}$. By using similar arguments as those in the proof of Claim 4.3 within the proof of Proposition 4.1, we obtain that $R(\Lambda_{\bar{r}}, X_2) \cong \mathbb{k}[[t]]$, which proves Proposition 4.6. \square

REFERENCES

1. M. Auslander, I. Reiten, and S. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, 1995.
2. D. J. Benson, *Representations and cohomology i: Basic representation theory of groups and associative algebras*, Cambridge Studies in Advanced Mathematics 30, Cambridge University Press, 1991.
3. F. M. Bleher, *Universal deformation rings of dihedral defect groups*, Trans. Amer. Math. Soc **361** (2009), 3661–3705.
4. F. M. Bleher and T. Chinburg, *Universal deformation rings and cyclic blocks*, Math. Ann. **318** (2000), 805–836.
5. ———, *Universal deformation rings need not be complete intersections*, Math. Ann. **337** (2007), 739–767.
6. F. M. Bleher, T. Chinburg, and B. de Smith, *Inverse problems for deformation rings*, Trans. Amer. Math. Soc (2012), in press.
7. F. M. Bleher and J. B. Froelich, *Universal deformation rings for the symmetric group S_5 and one of its double covers*, J. Pure Appl. Algebra **215** (2011), 523–530.
8. F. M. Bleher and G. Llorent, *Universal deformation rings for the symmetric group S_4* , Algebr. Represent. Theory **13** (2010), 255–270.
9. F. M. Bleher, G. Llorent, and J. B. Schaefer, *Universal deformation rings and dihedral blocks with two simple modules*, J. Algebra **345** (2011), 49–71.
10. F. M. Bleher and J. A. Vélez-Marulanda, *Universal deformation rings of modules over Frobenius algebras*, J. Algebra **367** (2012), 176–202.
11. M. C. R. Butler and C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra **15** (1987), 145–179.
12. C. W. Curtis and I. Reiner, *Methods of representation theory with applications to finite groups and orders*, vol. I, John Wiley and Sons, New York, 1981.
13. K. Erdmann, *Blocks of tame representation type and related algebras*, Lectures Notes in Mathematics 1428, Springer-Verlag, 1990.
14. R. Ile, *Change of rings in deformation theory of modules*, Trans. Amer. Math. Soc **356** (2004), 4873–4896.
15. H. Krause, *Maps between tree and band modules*, J. Algebra **137** (1991), 186–194.
16. B. Mazur, *An introduction to the deformation theory of Galois representations*, Modular Forms and Fermat’s Last Theorem, Springer-Verlag, Boston, MA, 1997, pp. 243–311.
17. M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. **130** (1968), 208–222.
18. D. Yau, *Deformation theory of modules*, Comm. Algebra **33** (2005), 2351–2359.

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